Unconstrained 0–1 Nonlinear Programming: A Nondifferentiable Approach

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(Received: 24 September 1990; accepted: 2 October 1991)

Abstract. The purpose of this paper is to give new formulations for the unconstrained 0–1 nonlinear problem. The unconstrained 0–1 nonlinear problem is reduced to nonlinear continuous problems where the objective functions are piecewise linear. In the first formulation, the objective function is a difference of two convex functions while the other formulations lead to concave problems. It is shown that the concave problems we obtain have fewer integer local minima than has the classical concave formulation of the 0–1 unconstrained 0–1 nonlinear problem.

Key words. 0–1 nonlinear programming, concave programming, piecewise linear functions and local minimum.

1. Introduction

A 0–1 unconstrained nonlinear problem (P) is generally formulated as

$$(P) \quad \mathrm{Min}\left\{\sum_{i=1}^{m} c_i\left(\prod_{j\in N_i} x_j\right) / x \in \{0,1\}^n\right\},$$

where $N_i \subset \{1, \ldots, n\}$, $c_i \in \mathbb{R}$, $m \in \mathbb{N}$.

In a very interesting paper, Rosenberg [8] showed that the problem (P) can be reduced to the continuous nonlinear problem

$$(\bar{P}) \quad \mathrm{Min}\left\{\sum_{i=1}^{m} c_i\left(\prod_{j\in N_i} x_j\right) / x \in [0,1]^n\right\},$$

since the optimal value of $(\bar{P})$ is attained in at least one vertex of the hypercube $[0, 1]^n$. Unfortunately, the objective function of $(\bar{P})$ is neither convex nor concave so that $(\bar{P})$ is as difficult as (P). Nevertheless, as discussed by Hansen–Jaumard–Mathon [4], the objective function of $(\bar{P})$ can be approximated by a difference of convex functions and therefore d-c programming (minimization of a difference of convex functions, see [9]) can be used. However, d-c programming can only solve small programs.

According to Hansen–Jaumard–Mathon, concave programming (minimization of a concave function over a convex set, see [6]) seems to be more promising. The transformation of (P) into a concave problem is quite classical (at least in the quadratic case, see [6]) and will be discussed in the next section.

The purpose of this paper is to give new formulations of (P) where the objective functions are piecewise linear. In the first formulation, the objective function is exactly a difference of two (piecewise linear) convex functions while the two last formulations lead to concave problems. As we shall prove and it should be useful, the concave problems we obtain have fewer integer local minima than the concave problem deduced from (P) in a classical way.

In order to have a self-contained paper, we first recall and discuss the classical transformation of (P) into a concave problem.

2. The Classical Transformation of (P) into a Concave Problem

The classical transformation of (P) into a concave problem uses the fact that

$$x_j \in \{0, 1\} \Leftrightarrow x_j = x_j^2 .$$

Thus we can subtract a quadratic term and add a linear term to the objective function f of (P) without modifying the values on the vertices of $[0, 1]^n$. The new objective function is then:

$$f_\rho(x) = f(x) - \frac{\rho}{2} \sum_{j=1}^{n} x_j^2 + \frac{\rho}{2} \sum_{j=1}^{n} x_j .$$

The value of ρ must be chosen so that f_ρ is concave, that is, so that for any x in $[0, 1]^n$, the greatest eigenvalue $\lambda_\rho(x)$ of $\nabla^2 f_\rho(x)$ is negative.

Since $\nabla^2 f_\rho(x) = \nabla^2 f(x) - \rho I \; \forall x \in [0, 1]^n$, we have $\lambda_\rho(x) = \lambda(x) - \rho$, where $\lambda(x)$ is the greatest eigenvalue of $\nabla^2 f(x)$.

In order that f be concave, we must take

$$\rho \geqslant \lambda(x) \quad \forall x \in [0, 1]^n \Leftrightarrow \rho \geqslant \bar{\lambda} = \max\{\lambda(x) / x \in [0, 1]^n\} .$$

The optimization problem as formulated above is harder than the initial problem (P), except if $\lambda(x)$ is constant as in the case where (P) is a 0–1 quadratic problem (but, even in this case, this value might be expensive to compute). It then seems better to use another technique: a matrix with a dominated diagonal is negative semi-definite (see [1]). Therefore we can consider the following objective function:

$$g(x) = f(x) - \frac{1}{2} \sum_{j=1}^{n} \left(\sum_{i=1}^{m} \delta_{ij} \,|\, c_i \,|\right) x_j^2 + \frac{1}{2} \sum_{j=1}^{n} \left(\sum_{i=1}^{m} \delta_{ij} \,|\, c_i \,|\right) x_j ,$$

$$\text{where} \quad \delta_{ij} = \begin{cases} 1 & \text{if } j \in N_i \text{ and } |\, N_i \,| \geqslant 2 , \\ 0 & \text{otherwise} . \end{cases}$$

The function g is then concave since, obviously, for any x in $[0, 1]^n$, $\nabla^2 g(x)$ has a dominated diagonal.

Thus minimizing $f_{\bar{\lambda}}$ or g over $[0, 1]^n$ is equivalent to solving (P). Some algorithms for concave programming can be found in [6]. The difficulty of a concave problem increases with the number of local minima. As mentioned before, we shall present other concave formulations of problem (P) but with a smaller number of integer minima than in the classical formulation.

Let us first express (P) as a d-c problem (a problem where the objective function is a difference of two convex functions).

3. Unconstrained 0–1 Nonlinear Programming as d-c Programming

In this paper, we shall often use the following lemma, which is easy to verify and which will allow us to express the objective function of (P) as a piecewise linear function.

LEMMA 3.1.

$$x \in \{0, 1\}^n \Rightarrow \forall M \subset \{1, \ldots, n\} \prod_{j \in M} x_j = \min_{j \in M}\{x_j\} \tag{3.1}$$

$$= \max\left\{0, \sum_{j \in M} x_j - |M| + 1\right\}. \tag{3.2}$$

Changing $\Pi_{j \in M} x_j$ by either (3.1) or (3.2), we obtain equivalent formulations for (P) with piecewise linear objective functions.

LEMMA 3.2.

$$\forall M \subset \{1, \ldots, n\}\ x \mapsto \min_{j \in M}\{x_j\} \quad \text{is a concave function},$$

$$\forall M \subset \{1, \ldots, n\}\ x \mapsto \max\left\{0, \sum_{j \in M} x_j - |M| + 1\right\} \quad \text{is a convex function}.$$

Proof. trivial. □

Let us now consider the following problem:

$$(P') \quad \mathrm{Min}\left\{\sum_{i=1}^{m} c_i \min_{j \in N_i}\{x_j\} / x \in \{0, 1\}^n\right\}.$$

By Lemma 3.1, this problem is equivalent to (P). Relaxing the integrality constraints in (P'), we obtain the continuous problem

$$(\mathrm{DCP}) \quad \mathrm{Min}\left\{\sum_{i=1}^{m} c_i \min_{j \in N_i}\{x_j\} / x \in [0, 1]^n\right\}.$$

By Lemma 3.2 and since coefficients c_i are unrestricted in sign, the objective function of (DCP) is a difference of two convex piecewise linear functions.

Theorem 3.3 below gives an equivalence between (P) and (DCP).

THEOREM 3.3. *The optimal value of (DCP) is attained in at least one vertex of the hypercube $[0, 1]^n$.*

Proof. Let x^* be an optimal solution of (DCP). Without loss of generality, we can suppose that $x_1^* \leqslant x_2^* \leqslant \cdots \leqslant x_n^*$. Thus, x^* is also an optimal solution of:

$$\operatorname{Min}\left\{\sum_{i=1}^{m} c_i \min_{j \in N_i}\{x_j\} \,/\, 0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant 1\right\}.$$

Let us denote by j_i the smallest j in N_i. Then, for any x feasible, $\min_{j \in N_i}\{x_j\} = x_{j_i}$ and x^* is also optimal solution of

$$\operatorname{Min}\left\{\sum_{i=1}^{m} c_{j_i} x_{j_i} \,/\, 0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant 1\right\}.$$

This last problem is linear and reaches its optimal value at an extreme point $\bar{x}$ of its feasible set S:

$$S = \{x \in \mathbb{R}^n / Hx \leqslant e\},$$

where H and e are the following matrix and vector

$$\begin{bmatrix} -1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 1 & -1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & & 1 & -1 & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 0 & 1 & -1 & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & -1 \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 1 \end{bmatrix}. \qquad e = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

H is totally unimodular since it represents the incidence matrix of the directed graph (see [3]):

Since e is an integer vector and H totally unimodular, all the extreme points of S are integer.

Thus, there exists $\bar{x} \in \{0, 1\}^n$ such that $0 \leqslant \bar{x}_1 \leqslant \bar{x}_2 \leqslant \cdots \leqslant \bar{x}_n$ and:

$$\sum_{i=1}^{m} c_{j_i} x_{j_i}^* = \sum_{i=1}^{m} c_{j_i} \bar{x}_{j_i},$$

that is $\bar{x}$ is an integer optimal solution of (DCP). □

Therefore the 0–1 unconstrained nonlinear problem appears to be a particular case of d-c programming (see [9] for algorithms).

Note that if c_i is non-positive for all N_i such that $|N_i| \geqslant 2$, then (DCP) (and therefore (P)), can be reduced to a convex minimization problem. This confirms that such problems (P), which have non-positive coefficients for the nonlinear terms, are easier to solve since their underlying convex structure is helpful for minimizing. For example, the quadratic 0–1 problem with negative double product coefficients is solvable in a polynomial time algorithm (see [7]).

On the other hand, if c_i is non-negative for all N_i such that $|N_i| \geq 2$, then (DCP) (and therefore (P)), is a concave minimization problem. In the next section, we shall give concave formulations for (P) even if there exists some i so that $|N_i| \geq 2$ and $c_i \leq 0$.

Note also that if we replace $\Pi_{j \in N_i} x_j$ by (3.2) we obtain a d-c objective function and an equivalent formulation (P'') for (P), but we are not able to relax the integrality constraints in (P'') without modifying the optimal value . . .

4. 0–1 Unconstrained Nonlinear Programming as Concave Programming

In order to obtain concave formulations for (P), we have to eliminate the convex part of (DCP), that is, we have to reformulate $\min_{j \in N_i} \{x_j\}$ for i such that $c_i \leq 0$ and $|N_i| \geq 2$. For the sake of simplicity suppose that the set of such i is $\{1, \ldots, p\}$.

The first possibility is to linearize the convex part. This can be done by introducing new variables z, and several constraints in (DCP). We then obtain a semi-linearized concave problem.

$$
\begin{aligned}
(SLCP) \quad \text{Min} \quad & \sum_{i=1}^{p} c_i z_i + \sum_{i=p+1}^{m} c_i \min_{j \in N_i}\{x_j\} & \\
\text{s.t.} \quad & z_i \leq x_j & \forall j \in N_i,\ \forall i \in \{1, \ldots, p\}\,. \\
& x \in [0, 1]^n & \\
& z \geq 0 &
\end{aligned}
$$

Obviously, the objective function of $(SLCP)$ is concave. The following theorem establishes the equivalence, via Theorem 3.3, between $(SLCP)$ and (P).

THEOREM 4.1. *$(SLCP)$ and (DCP) are two versions of the same problem.*

Proof. It is sufficient to prove that $(\bar{x}, \bar{z})$ optimal solution of $(SLCP) \Rightarrow \bar{z}_i = \min_{j \in N_i} \{x_j\}$. Suppose this is not true. Then, there exists $\bar{i} \in \{1, \ldots, p\}$ so that $\bar{z}_{\bar{i}} < \min_{j \in N_{\bar{i}}} \{\bar{x}_j\}$. Consider now (x^*, z^*) defined as:

$$
\begin{aligned}
& x^* = \bar{x} & \\
& z_i^* = \bar{z}_i & \forall i \neq \bar{i}\,. \\
& z_{\bar{i}}^* = \min_{j \in N_{\bar{i}}} \{\bar{x}_j\} &
\end{aligned}
$$

Since $c_{\bar{i}} < 0$ and $\bar{z}_{\bar{i}} < z_{\bar{i}}^*$, we have:

$$
\sum_{i=1}^{p} c_i z_i^* + \sum_{i=p+1}^{m} \min_{j \in N_i} \{x_j^*\} < \sum_{i=1}^{p} c_i \bar{z}_i + \sum_{i=p+1}^{m} \min_{j \in N_i} \{\bar{x}_j\}
$$

and this is a contradiction since $(\bar{x}, \bar{z})$ is optimal. □

Therefore, the 0–1 unconstrained nonlinear problem (P) can be reduced to a concave minimization problem where the objective function is piecewise linear.

It is important to note that if all the coefficients of the nonlinear terms are non-positive then $(SCLP)$ is a linear problem and, therefore, (P) appears to be polynomially (in n and m) solvable.

$(SCLP)$ has the drawback of introducing additional variables and constraints. We can avoid this inconvenience: changing $\Pi_{j\in M} x_j$ by $\max\{0, \Sigma_{j\in M} x_j - |M| + 1\}$ for i in $\{1, \dots, p\}$, we obtain a concave objective function and the following concave minimization problem (CMP).

$$(CMP)\quad \mathrm{Min}\left\{\sum_{i=1}^{p} c_i \max\left\{0, \sum_{j\in N_i} x_j - |N_i| + 1\right\} + \sum_{i=p+1}^{m} c_i \min_{j\in N_i}\{x_j\} \,/\, x \in [0,1]^n\right\}.$$

Obviously, (CMP) is equivalent to (P) since a concave minimization problem reaches its optimal value at a vertex of its feasible set.

The main advantage of (DCP), $(SLCP)$ and (CMP) over the classical transformation of (P) into a concave minimization problem is that they have fewer integer local minima. First, recall that the only practical classical way of transforming (P) into a concave problem is to consider the objective function (see Section 2):

$$g(x) = f(x) - \frac{1}{2}\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\delta_{ij}|c_i|\right)x_j^2 + \frac{1}{2}\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\delta_{ij}|c_i|\right)x_j$$

where $\delta_{ij} = \begin{cases} 1 & \text{if } j \in N_i \text{ and } |N_i| \geq 2 \\ 0 & \text{otherwise} \end{cases}$

and the optimization problem (Q)

$$(Q)\quad \mathrm{Min}\{g(x)/x \in [0,1]^n\}.$$

In our proof, we shall need the following lemma.

LEMMA 4.2. *For any* $M \subset \{1, \dots, p\}$ *and for any* x *in* $[0,1]^n$

(i) $$\max\left\{0, \sum_{j\in M} x_j - |M| + 1\right\} \leq \min_{j\in M}\{x_j\},$$

(ii) $$\max\left\{0, \sum_{j\in M} x_j - |M| + 1\right\} \geq \frac{1}{2}\sum_{j\in M} x_j^2 - \frac{1}{2}\sum_{j\in M} x_j + \prod_{j\in M} x_j,$$

(iii) $$\min_{j\in M}\{x_j\} \leq \frac{1}{2}\sum_{j\in M} x_j^2 - \frac{1}{2}\sum_{j\in M} x_j + \prod_{j\in M} x_j.$$

Proof. (i) If $\max\{0, \Sigma_{j\in M} x_j - |M| + 1\} = 0$ then the result is obvious. Otherwise, $\max\{0, \Sigma_{j\in M} x_j - |M| + 1\} = x_{\bar{j}} + \Sigma_{j\in M-\{\bar{j}\}} x_j - |M| + 1$, where $x_{\bar{j}} = \min_{j\in M}\{x_j\}$ but $\Sigma_{j\in M-\{\bar{j}\}} x_j - |M| + 1 \leq 0 \Rightarrow \max\{0, \Sigma_{j\in M} x_j - |M| + 1\} \leq x_{\bar{j}} = \min_{j\in M}\{x_j\}$.

(ii) Consider the optimization problem

$$\max\left\{\frac{1}{2}\sum_{j\in M} x_j^2 - \frac{1}{2}\sum_{j\in M} x_j + \prod_{j\in M} x_j - \max\left\{0, \sum_{j\in M} x_j - |M| + 1\right\} \Big/ x \in [0,1]^n\right\}.$$

Its optimal value is equal to $\max\{v_1, v_2\}$

$$v_1 = \max\left\{\frac{1}{2}\sum_{j\in M} x_j^2 - \frac{1}{2}\sum_{j\in M} x_j + \prod_{j\in M} x_j \Big/ \sum_{j\in M} x_j \leq |M| - 1; x \in [0,1]^n\right\},$$

$$v_2 = \max\left\{\frac{1}{2}\sum_{j\in M} x_j^2 - \frac{3}{2}\sum_{j\in M} x_j + \prod_{j\in M} x_j + |M| - 1 \Big/ \sum_{j\in M} x_j \geq |M| - 1; x \in [0\ 1]^n\right\}.$$

Both objective functions are convex (since, for any x in $[1,0]^n$, the Hessian has a dominating diagonal). Thus, the optimal values are attained at extreme points of the feasible sets.

Since a matrix composed of 0–1 entries and having only one row is totally unimodular, all the extreme points of the feasible sets are integer.

Note that if $x \in \{0,1\}^n$ is feasible for the first problem then $\Pi_{j\in M} x_j = 0$. Thus $v_1 = 0$ since $x_j \in \{0,1\} \Leftrightarrow x_j = x_j^2$.

$x = (1, \ldots, 1)$ is feasible for the second problem and the corresponding value is 0. For all other integer points, we have $\Pi_{j\in M} x_j = 0$ and $\Sigma_{j\in M} x_j - |M| + 1 = 0$. Consequently, $v_2 = 0$.

Therefore:

$$\forall x \in [0,1]^n \ \frac{1}{2}\sum_{j\in M} x_j^2 - \frac{1}{2}\sum_{j\in M} x_j + \prod_{j\in M} x_j \leq \max\left\{0, \sum_{j\in M} x_j - |M| + 1\right\}.$$

(iii) Consider now the optimization problem:

$$\max\left\{\frac{1}{2}\sum_{j\in M} x_j^2 - \frac{1}{2}\sum_{j\in M} x_j + \prod_{j\in M} x_j - \min_{j\in M}\{x_j\} / x \in [0,1]^n\right\}$$

and let x^* be an optimal solution of this problem. Without loss of generality we can suppose that $x_1^* \leq x_2^* \leq \cdots \leq x_n^*$ and $1 \in M$.

Thus x^* is also an optimal solution of

$$\max\left\{\frac{1}{2}\sum_{j\in M} x_j^2 - \frac{1}{2}\sum_{j\in M} x_j + \prod_{j\in M} x_j - x_1 / 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1\right\}.$$

Since the objective function is convex, the optimal value is reached at an extreme point of its feasible set. It was shown, in the proof of Theorem 3.3, that such points are integer. Therefore, there exists $\bar{x} \in \{0,1\}^n$ so that $\bar{x}$ is an optimal solution of the above problem.

Thus: $0 \leq \bar{x}_1 \leq \bar{x}_2 \leq \cdots \leq \bar{x}_n \leq 1$.

If $\bar{x}_1 = 0$ then $\Pi_{j\in M} \bar{x}_j = 0$ and the corresponding objective value is 0. If $\bar{x}_1 = 1$

then $\Pi_{j\in M}\bar{x}_j = 1$ and the corresponding objective value is also 0. So that:

$$\min_{j\in M}\{x_j\} \leq \frac{1}{2}\sum_{j\in M} x_j^2 - \frac{1}{2}\sum_{j\in M} + \prod_{j\in M} x_j \quad \forall x\in[0,1]^n . \qquad \square$$

We can now establish some relations between the local minima of (DCP), $(SLCP)$, (CMP) and (Q).

THEOREM 4.3. (i) *x local minimum for $(DCP) \Rightarrow (x, z)$ local minimum for $(SLCP)$ with $z_i = \min_{j\in N_i}\{x_j\}$ $\forall i\in\{1,\ldots,p\}$.*
(ii) (x, z) local minimum for $(SLCP) \Rightarrow x$ local minimum for (DCP).
(iii) x local minimum for (DCP) and x integer $\Rightarrow x$ local minimum for (CMP).
(iv) x local minimum for (CMP) and x integer $\Rightarrow x$ local minimum for (Q).

Proof. (i) and (ii) are trivial.

(iii) For the sake of simplicity, let us define

$D(x) = \sum_{i=1}^{m} c_i \min_{j\in N_i}\{x_j\}$ as the objective function of (DCP).

$M(x) = \sum_{i=1}^{p} c_i \max\left\{0, \sum_{j\in N_i} x_j - |N_i| + 1\right\} + \sum_{i=p+1}^{m} c_i \min_{j\in N_i}\{x_j\}$ as the objective function of (CMP).

$g(x) = \sum_{i=1}^{m} c_i \prod_{j\in N_i} x_j + \frac{1}{2}\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\delta_{ij}|c_i|\right)(x_j - x_j^2)$ as the objective function of (Q).

Since x is a local minimum for (DCP), there exists an open set V so that $x\in V$ and so that:

$$\forall y\in V\cap[0,1]^n \quad D(x)\leq D(y) .$$

By Lemma 4.2 we have:

$$\forall y\in V\cap[0,1]^n \quad D(y)\leq M(y) ,$$

but x integer $\Rightarrow D(x) = M(x)$. Thus, there exists an open set V so that $x\in V$ and so that:

$$\forall y\in V\cap[0,1]^n \quad M(x) = D(x)\leq D(y)\leq M(y) ,$$

which means that x is a local minimum for (CMP).

(iv) x is local minimum for (CMP) implies that there exists an open set V such that $x\in V$ and

$$\forall y\in V\cap[0,1]^n \quad M(x)\leq M(y) .$$

By Lemma 4.2 we have:

$$\forall y\in V\cap[0,1]^n \quad M(y)\leq g(y)$$

but x integer $\Rightarrow M(x) = g(x)$. Thus, there exists an open set V so that $x \in V$ and so that:

$$\forall y \in V \cap [0, 1]^n \quad M(x) = g(x) \leq M(y) \leq g(y),$$

which means that x is a local minimum for (Q). □

Therefore, the classical transformation of (P) into a concave problem might have more integer local minima than (DCP), $(SLCP)$ and (CMP). Note that the implications (iii) and (iv) do not apply to local minima which are not integer. However, we are not really interested in non-integer local minima since, in solving (P), we are looking for local minima which are extreme points of the feasible set. Furthermore, if there exists a non-integer local minimum, then there also exists an integer local minimum having the same objective value.

Defining $I(DCP)$ (respectively $I(SLCP)$, $I(CMP)$, $I(Q)$) as the set of integer local minima of (DCP) (respectively $(SLCP), \ldots, (Q)$), from Theorem 4.3, we can deduce:

$$I(SLCP) = I(DCP) \subset I(CMP) \subset I(Q).$$

One might wonder if the inverse inclusions are valid. The example below gives the answer to such a question: the inverse inclusions are definitely not valid.

EXAMPLE 4.4. Consider the 0–1 unconstrained nonlinear problem

$$(P) \quad \mathrm{Min}\{-4x_1x_2x_3 + x_1x_2 + x_1x_3 + x_2x_3 / x \in \{0, 1\}^3\},$$

then (Q), (DCP) and (P) are

$$(Q) \quad \mathrm{Min}\{g(x) / x \in [0, 1]^3\}$$

$$g(x) = -4x_1x_2x_3 + x_1x_2 + x_1x_3 + x_2x_3 + 3(x_1 + x_2 + x_3) - 3(x_1^2 + x_2^2 + x_3^2),$$

$$(DCP) \quad \mathrm{Min}\{D(x) / x \in [0, 1]^3\}$$

$$D(x) = -4\min\{x_1, x_2, x_3\} + \min\{x_1, x_2\} + \min\{x_1, x_3\} + \min\{x_2, x_3\},$$

$$(CMP) \quad \mathrm{Min}\{M(x) / x \in [0, 1]^3\}$$

$$M(x) = \min\{x_1, x_2\} + \min\{x_1, x_3\} + \min\{x_2, x_3\} - 4\max\{0, \Sigma_{i=1}^3 x_i - 2\}.$$

Let us examine the integer points in order to determine whether they are local minima for one of the continuous problems above.

- $\bar{x} = (0, 0, 0)$ is a local minimum for (CMP), therefore also for (Q), but it is not a local minimum for (DCP) since:
  - $D((0, 0, 0) + t(1, 1, 1)) = -t \Rightarrow (1, 1, 1)$ is a feasible descent direction at $(0, 0, 0)$ for (DCP).
  - $M((0, 0, 0) + t(d_1, d_2, d_3)) = t(\min\{d_1, d_2\} + \min\{d_1, d_3\} + \min\{d_2, d_3\})$ for

t sufficiently small, but d feasible direction at $(0,0,0) \Rightarrow d_1 \geq 0$, $d_2 \geq 0$, $d_3 \geq 0$. Thus any feasible direction for (CMP) is an ascent direction.

- $\bar{x} = (1,0,0)$ is a local minimum for (CMP), therefore for (Q), but is not a local minimum for (DCP).
  - $d = (0,1,1)$ is a feasible descent direction at $(1,0,0)$ for (DCP).
  - $M((1,0,0) + t(d_1, d_2, d_3)) = t(d_2 + d_3 + \min\{d_2, d_3\})$ for t sufficiently small. As any feasible direction is such that $d_2 \geq 0$, $d_3 \geq 0$, any feasible direction is an ascent direction for (CMP).
- By symmetry $(0,1,0)$ and $(0,0,1)$ are local minima for (CMP) and (Q), but not for (DCP).
- $\bar{x} = (1,1,0)$ is not a local minimum for (CMP), therefore not for (DCP), but it is a local minimum for (Q).
  - $M((1,1,1) + t(0,1,1)) = -2t \Rightarrow (0,1,1)$ is a feasible descent direction for (CMP).
  - $\langle \nabla g(1,1,0), d\rangle = -2d_1 - 2d_2 + 3d_3$ but d feasible for $(1,1,0) \Rightarrow d_1 \leq 0$, $d_2 \leq 0$, $d_3 \geq 0$. Thus $d \neq 0$ and feasible $\Rightarrow \langle \nabla g(1,1,0), d\rangle > 0$ and then any feasible direction is an ascent direction for (Q).
- By symmetry $(1,0,1)$ and $(0,1,1)$ are not local minima for (CMP), nor for (DCP), but are local minima for (Q).
- $\bar{x} = (1,1,1)$ is a local minimum for all the problems since it is the optimal solution.

Then we have:

$$
\begin{aligned}
I(Q) \quad &= \{0,1\}^3 \\
I(CMP) &= \{(0,0,0);\ (1,0,0);\ (0,1,0);\ (0,0,1);\ (1,1,1)\} \\
I(DCP) &= I(SLCP) = \{(1,1,1)\}.
\end{aligned}
$$

For this particular problem (P), all the integer points are local minima for (Q) but only a subset of them are local minima for (CMP) or (DCP). Furthermore, we must emphasize that $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ are not strict local minima for (CMP) (that is we can find a feasible direction d so that M is constant on the corresponding line) while they are strict local minima for (Q).

5. Conclusion

If, as we believe, the number of local minima is a criterion for classifying concave formulations of (P), then it appears that $(SLCP)$ and (CMP) are really better than (Q). As shown by Example 4.4, the number of integer local minima could be strongly reduced by using $(SLCP)$ instead of the classical transformation of (P) into a concave minimization problem. On the other hand, it seems that concave minimization problems with piecewise linear objective functions are tractable (see [6]).

We emphasize that a 0–1 unconstrained nonlinear problem with only negative coefficients for the nonlinear terms does not need to be formulated as a concave problem. This particular kind of problem can be reduced to a linear problem and is then polynomially (in n and m) solvable.

In closing, we note the formulations suggested for the 0–1 unconstrained nonlinear problem can be extended to 0–1 nonlinear problems with linear constraints since Kalantari–Rosen [5] and Borchardt [2] showed how a 0–1 nonlinear minimization problem with linear constraints and a concave objective function can be modified in order to obtain an integer solution when the integrality constraints are dropped.

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